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1993 J. Phys. A: Math. Gen. 26 4035

(http://iopscience.iop.org/0305-4470/26/16/019)

J. Phys. A: Math. Gen. 26 (1993) 4035–4046. Printed in the UK

The algebraic structure of generalized Ermakov systems in three dimensions

K S Govinder†, C Athorne‡ and P G L Leach†§

† Department of Mathematics and Applied Mathematics, University of Natal, King George V Ave, Durban, R S A, 4001, South Africa

‡ Department of Mathematics, University of Glasgow, University Gardens, Glasgow, G12 8QW, Scotland, UK

§ Associate member of the Centre for Nonlinear Studies, University of the Witwatersrand, Johannesburg, South Africa

Received 9 February 1993

Abstract. The characteristic algebra of generalized Ermakov systems is $sl(2,\mathbb{R})$. The structure of these systems in three dimensions is obtained. A subset in the form of an equation of motion with the additional requirement of $so(3)$ symmetry is studied. It includes the classical equation of the magnetic monopole. The existence of three vectors of Poincaré type is established. Consideration is given to weak generalized Ermakov systems in which the symmetry breaking occurs in the radial equation.

1. Introduction

Since Ray and Reid (1979) revealed them to the Western literature, Ermakov systems and their generalizations have attracted wide attention. The Ermakov system (Ermakov 1880) is the combination of the equation for a time-dependent oscillator

$$\ddot{q} + \omega^2(t)q = 0 \tag{1.1}$$

and an auxiliary equation, generally known as the Pinney equation (Pinney 1950),

$$\ddot{\rho}^2 + \omega^2(t)\rho = \rho^{-3}. \tag{1.2}$$

Elimination of $\omega^2(t)$ between (1.1) and (1.2), the introduction of an integrating factor, $\rho\dot{q} - \dot{\rho}q$, and integration produces the first integral

$$I = \tfrac{1}{2}[(\rho\dot{q} - \dot{\rho}q)^2 + (q/\rho)^2] \tag{1.3}$$

which is usually called the Lewis invariant after H Ralph Lewis who 'rediscovered' it in 1966 (Lewis 1967, 1968) in an application of Kruskal's asymptotic method (Kruskal 1962).

The generalized Ermakov system of Ray and Reid (1979)

$$\ddot{x} + \omega^2(t)x = x^{-3}f(y/x) \tag{1.4}$$

$$\ddot{y} + \omega^2(t)y = y^{-3}g(y/x) \tag{1.5}$$

where f and g are arbitrary functions of their arguments, has a first integral obtained in the same way as that for (1.1) and (1.2) which is

$$I = \tfrac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)]\,\mathrm{d}u. \tag{1.6}$$

The Ermakov invariant (1.6) persists if $\omega^2(t)$ is replaced by *anything* (Ray 1980, Ray and Reid 1980, Goedert 1990, Leach 1991).

Generalized Ermakov systems have been widely treated and we refer the reader to Leach (1991) for references. Our particular interest in this paper is the relationship between the structure of the differential equations of a three-dimensional generalized Ermakov system and its underlying Lie algebraic structure. In part this has been motivated by the observation of Leach (1991) that the nature of generalized Ermakov systems is explained in terms of the Lie symmetry algebra $sl(2, \mathbb{R})$ and is more obvious if plane polar coordinates (r, θ) are used. Then (1.4), (1.5) and (1.6) become

$$\ddot{r} - r\dot{\theta}^2 + \omega^2(t)r = r^{-3}F(\theta) \tag{1.7}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = r^{-3}G(\theta) \tag{1.8}$$

$$I = \tfrac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta)\,\mathrm{d}\theta. \tag{1.9}$$

The Ermakov–Lewis invariant (1.9) possesses the three Lie point symmetries

$$G_1 = \partial/\partial t \tag{1.10}$$

$$G_2 = 2t\partial/\partial t + r\partial/\partial r \tag{1.11}$$

$$G_3 = t^2\partial/\partial t + tr\partial/\partial r \tag{1.12}$$

with the Lie brackets

$$[G_1, G_2] = 2G_1 \qquad [G_1, G_3] = G_2 \qquad [G_2, G_3] = 2G_3 \tag{1.13}$$

which are the relations for the Lie algebra $sl(2, \mathbb{R})$. The invariant (1.9) is obtained by integration of the angular equation (1.8) which also possesses $sl(2, \mathbb{R})$ symmetry. With the representation (1.10)–(1.12) of $sl(2, \mathbb{R})$ used here this is not the case for (1.7) unless the $\omega^2(t)r$ term (or anything else) is absent. However, we do note that the $\omega^2(t)r$ term can be accommodated by a different set of operators which are related to (1.10)–(1.12) by means of a point transformation (Leach and Gorringe 1990). In Leach (1991) it was proposed that the expression *generalized Ermakov* system be restricted to systems for which the equations of motion possessed $sl(2, \mathbb{R})$ symmetry and that systems which had only the invariant be termed *weak* generalized Ermakov systems.

The observations reported by Leach (1991) were based on a Lie algebraic analysis of (1.4), (1.5) and (1.6). It was natural to ask what systems possessed $sl(2, \mathbb{R})$ symmetry. Govinder and Leach (1992) showed that the most general system of second-order ordinary differential equations in two dimensions invariant under the action of the $sl(2, \mathbb{R})$ representation (1.10)–(1.12) consisted of equations of the form

$$F(\theta, r^2\dot{\theta}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}) = 0. \tag{1.14}$$

(The extension to higher dimensions requires just the addition of the arguments ϕ, $r^2\dot{\phi}$, $r^4\ddot{\phi}+2r^3\dot{r}\dot{\phi}, \ldots$.) A subset of equations of the type (1.14) can be written in terms of the form of the Newtonian equation of motion of a particle as

$$\ddot{r} - r\dot{\theta}^2 = r^{-3} f(\theta, r^2\dot{\theta}) \tag{1.15}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = r^{-3} g(\theta, r^2\dot{\theta}). \tag{1.16}$$

Equations (1.15) and (1.16) take a simpler form if we introduce new time

$$T := \int r^{-2}\,\mathrm{d}t \tag{1.17}$$

and the inverse radial distance $\chi = r^{-1}$. They become

$$\chi'' + [\theta'^2 + f(\theta, \theta')]\chi = 0 \tag{1.18}$$

$$\theta'' = g(\theta, \theta') \tag{1.19}$$

where $' := (\mathrm{d}/\mathrm{d}T)$. There is a local solution of (1.19), $\theta = \theta(T, \theta_0, \theta_0')$, and, when this is substituted into (1.18) together with $\theta' = \theta'(T, \theta_0, \theta_0')$, (1.18) is reduced to to the equation for a two-parameter family of time-dependent harmonic oscillators, the family of general solutions of which will contain four parameters. Under suitable conditions on $g(\theta, \theta')$ (1.19) can be integrated to give the first integral

$$I = M(\theta, \theta') \tag{1.20}$$

which is the generalization of the Ermakov–Lewis invariant (1.9). Again under suitable conditions (1.20) can be integrated to give a second first integral

$$J = T - \int \frac{\mathrm{d}\theta}{N(\theta, I)} \tag{1.21}$$

where $N(\theta, I)$ is obtained by the inversion of (1.20). In general this will not be possible as $M(\theta, \theta')$ will be only locally defined (for example on open neighbourhoods of analytic points of g in (θ, θ') space) and typically $N(\theta, I)$ will be infinitely branched. In the case that f and g in (1.18) and (1.19) are free of θ' we obtain from (1.19) the usual expression for the Ermakov–Lewis invariant

$$I = \tfrac{1}{2}\theta'^2 - \int g(\theta)\,\mathrm{d}\theta \tag{1.22}$$

and (1.21) becomes

$$J = T - \int \mathrm{d}\theta \Big/ \left\{ 2\left[I + \int^{\theta} g(s)\,\mathrm{d}s \right] \right\}^{1/2}. \tag{1.23}$$

We note that, if we impose the additional requirement that equation (1.14) (and so (1.18) and (1.19)) be invariant under the action of the rotation group in two dimensions with generator $\partial/\partial\theta$ so that the algebra is $sl(2, \mathbb{R}) \oplus so(2)$, (1.19) is now

$$\theta'' = g(\theta'). \tag{1.24}$$

The Ermakov–Lewis invariant is given by

$$I = \int \frac{\theta' \mathrm{d}\theta'}{g(\theta')} - \theta \tag{1.25}$$

and an explicitly time-dependent integral is given by

$$K = T - \int \frac{\mathrm{d}\theta'}{g(\theta')}. \tag{1.26}$$

The integral (1.23) comes from the elimination of θ' between (1.25) and (1.26).

In this paper we extend the consideration of systems of differential equations invariant under $sl(2,\mathbb{R})$ to three dimensions. We have already noted that the generalization of (1.14) to higher dimensions is trivial. However, we find that the imposition of rotational invariance by making the invariance algebra $sl(2,\mathbb{R}) \oplus so(3)$ yields an interesting class of differential equations which includes the classical equation for the magnetic monopole. The invariance of this equation under the elements of the algebra $sl(2,\mathbb{R}) \oplus so(3)$ has already been reported by Moreira *et al* (1985) (although they preferred to use the isomorphic algebra $so(2,1) \oplus so(3)$). The monopole is also known to possess a conserved vector called Poincaré's vector (Poincaré 1896). We shall see that the general system to be discussed here possesses three such vectors and that the solution of the system of equations reduces to the determination of the three Poincaré vectors and the solution of the radial equation corresponding to (1.18). We should point out that, in the case of the monopole, the vector usually referred to as Poincaré's vector is obtained by elementary vectorial manipulation of the equation of motion. The derivation of the two other vectors which, because of their nature, we also term Poincaré vectors is by no means transparent even in this simple case. We also consider weak generalized Ermakov systems in three dimensions.

2. Equations invariant under $sl(2,\mathbb{R}) \oplus so(3)$

In spherical polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$) the equation corresponding to (1.14) invariant under the representation (1.10)–(1.12) of $sl(2,\mathbb{R})$ is

$$F(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, r^3\ddot{r}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}) = 0. \tag{2.1}$$

To make reasonable sense as a system of second-order differential equations in three dependent variables we need a system of three equations of the form

$$r^3\ddot{r} = f(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \tag{2.2}$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = g(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}) \tag{2.3}$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = h(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}). \tag{2.4}$$

(One could conceive of variations on this. By way of example—not definitive nor intended to be exclusive—(2.4) could be replaced by

$$H(\theta, \phi, r^2\dot{\theta}, r^2\dot{\phi}, I) = 0 \tag{2.5}$$

where I is a parameter which may be taken to be the value of a first integral. If I has a particular value, I_0, in which case it could just as well be omitted from (2.5), we are in the realm of configuration invariants. To keep the discussion concise we do not digress into this specialized area. The reader is referred to Sarlet *et al* (1985) for a discussion of the relationship between systems of second-order equations, first integrals and configurational invariants.)

In terms of the new time T and inverse radial distance χ equations (2.2)–(2.4) are

$$\chi'' = -f(\theta, \phi, \theta', \phi')\chi \tag{2.6}$$

$$\theta'' = g(\theta, \phi, \theta', \phi') \tag{2.7}$$

$$\psi'' = h(\theta, \phi, \theta', \phi'). \tag{2.8}$$

In contrast to the pair of equations (1.18) and (1.19) for which (1.19) was 'in principle' integrable and so (1.18) reduced to the time-dependent oscillator, the situation with the system (2.6)–(2.8) is much more complex. Given θ and ϕ as functions of T, (2.6) is straightforward enough as it is linear in χ.

We confine our attention to systems for which, in addition to invariance under $sl(2, \mathbb{R})$ there is also rotational invariance, i.e. the system of equations is also invariant under the action of the generators of $so(3)$, namely

$$G_4 = \partial/\partial\phi \tag{2.9}$$

$$G_5 = \sin\phi\partial/\partial\theta + \cot\theta\cos\phi\partial/\partial\phi \tag{2.10}$$

$$G_6 = \cos\phi\partial/\partial\theta - \cot\theta\sin\phi\partial/\partial\phi. \tag{2.11}$$

We add this constraint from considerations of possible physical applications. The applications of $(\partial/\partial\phi)$ to (2.2)–(2.4) (alternatively (2.6)–(2.8), but the sequel suggests that the former should be used) is simple enough, f, g and h must be ϕ free. As the second extension of G_5 (æq G_6) mixes $\ddot{\theta}$ and $\ddot{\phi}$ terms, (2.3) and (2.4) must be treated as a coupled system whereas (2.2) can be treated by itself. The actual analysis involved makes for a pleasant and straightforward exercise. The action of $G_6^{[2]}$ makes no difference to the result which is to be expected since $G_6 = [G_4, G_5]$. (Given a symmetry $G = \tau\partial/\partial t + \eta_i\partial/\partial x_i$, the second extension is $G^{[2]} = G + (\dot{\eta}_i - \dot{x}_i\dot{\tau})\partial/\partial\dot{x}_i + (\ddot{\eta}_i - 2\ddot{x}_i\dot{\tau} - \dot{x}_i\ddot{\tau})\partial/\partial\ddot{x}_i$.) We find that the most general system of the form (2.2)–(2.4) invariant under $sl(2, \mathbb{R}) \oplus so(3)$ is

$$r^3\ddot{r} = A_1(L) \tag{2.12}$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = r^4\dot{\phi}^2\sin\theta\cos\theta + B(L)r^2\dot{\theta} - C(L)r^2\dot{\phi}\sin\theta \tag{2.13}$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = -2r^4\dot{\theta}\dot{\phi}\cot\theta + \frac{1}{\sin\theta}[B(L)r^2\dot{\phi}\sin\theta + C(L)r^2\dot{\theta}] \tag{2.14}$$

where A_1, B and C are arbitrary functions of their argument L, where

$$L^2 := r^4(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \tag{2.15}$$

is the square of the magnitude of the angular momentum. The three equations (2.12)–(2.14) may be written in the compact vectorial form

$$\ddot{\boldsymbol{r}} = \frac{1}{r^3}\{A(L)\hat{\boldsymbol{r}} + B(L)\hat{\boldsymbol{\omega}} + C(L)\hat{\boldsymbol{L}}\} \tag{2.16}$$

where we have replaced $A_1(L)$ by $A(L)+L^2$. In an obvious notation $\hat{r}$ and $\hat{L}$ are the unit vectors in the direction of the radius vector and the angular momentum vector $L(:=r\times\dot{r})$. The unit vector $\hat{\omega}:=\hat{L}\times\hat{r}$ is in the direction of the rate of change of $\hat{r}$ and is the natural generalization of $\hat{\theta}$ in plane polar coordinates.

In terms of the definition of generalized and weak generalized Ermakov systems (2.2)–(2.4) represents the three-dimensional form of the generalized Ermakov system. The addition of some extra term to (2.2) would be in the spirit of the meaning of weak generalized Ermakov system as given by Leach (1991). However, two points should be made. The first is that under suitable (for example analyticity) conditions (2.2)–(2.4) have integrals, i.e. constants of integration, defined over some local neighbourhood. The existence of one or more global first integrals for (2.3), (2.4) or a combination of (2.3) and (2.4) would require some constraints on the functions g and h. The second is that we have chosen the radial equation to be the one which leads to the symmetry breaking. It made sense in two dimensions as we were guaranteed the 'in principle' existence of an Ermakov–Lewis invariant provided that the system maintained $sl(2,\mathbb{R})$ symmetry in the radial equation. This is lost in the general three-dimensional case and further thought needs to be given to a correct terminology.

To conclude this section we make some observations about (2.16). For B and C zero and $A(L)$ a constant (L is conserved) we have the equation for a Newton–Cotes spiral (Whittaker 1944) which, in essence, is the free particle in the plane with an excess or deficit of angular momentum. For A and B zero and $C(L)$ proportional to L ($=\lambda L$) a constant (L is again conserved) we have the classic equation of a particle moving in the field of a magnetic monopole. In this case it is well known that there exists the first integral

$$P=L+\lambda\hat{r} \tag{2.17}$$

and the motion is on the surface of a cone of semi-vertex angle given by $\cos^{-1}(C/PL)$ (Poincaré 1896). It is only more recently that Moreira *et al* (1985) demonstrated that the algebra was $so(2,1)\oplus so(3)$ (isomorphic to $sl(2,\mathbb{R})\oplus so(3)$). We note that the classical monopole is a Hamiltonian system and the components of the Poincaré vector possess the algebra $so(3)$ under the operation of taking the Poisson bracket (Mladenov 1988).

3. Poincaré vector for (2.16)

The combination of the existence of the Poincaré vector (2.17) and the symmetry algebra $sl(2,\mathbb{R})\oplus so(3)$ for the classical monopole equation

$$\ddot{r}=C(L)\hat{L}/r^3 \tag{3.1}$$

suggests that it may be fruitful to look for a similar vector for the general equation (2.16). Elementary manipulation of (3.1) produces (2.17) more or less without trying. This is not the case with (2.16). However, an equally simple-minded approach does yield interesting results. We assume the existence of a vector of Poincaré type given by

$$P:=I\hat{r}+J\hat{\omega}+K\hat{L} \tag{3.2}$$

where I, J and K are functions to be determined. Requiring that $\dot{P}$ be zero when (2.16) is satisfied leads to the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} I\\ J\\ K\end{pmatrix}=r^{-2}\begin{pmatrix}0 & L & 0\\ -L & 0 & C/L\\ 0 & -C/L & 0\end{pmatrix}\begin{pmatrix} I\\ J\\ K\end{pmatrix} \tag{3.3}$$

which in terms of new time T are

$$\begin{pmatrix} I \\ J \\ K \end{pmatrix}' = r^{-2} \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} \begin{pmatrix} I \\ J \\ K \end{pmatrix}. \tag{3.4}$$

Equations (3.4) have a geometrical interpretation. They are the Serret–Frenet formulae associated with a curve of curvature L and torsion $C(L)/L$, parametrized by T. An orthonormal triad of solution vectors represents the principal triad of the curve, consisting of tangent, normal and binormal vectors.

As an aside we note that this approach is not feasible for the two-dimensional system of equations since $\dot{\hat{r}}$ and $\dot{\hat{\theta}}$ are then multiples of $\dot{\theta}$ and each multiple is a property of the geometry of the plane and is independent of the mechanics. The only way to make progress would be to specify the $\dot{r}$ and $\dot{\theta}$ dependence in $\boldsymbol{P}$. This has not been necessary in the present case because the dynamics is introduced via $\hat{\omega}$.

The scalar product of (2.16) with $\hat{r}$ is

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) = r^{-3} A(L) \tag{3.5}$$

which in terms of χ and T is

$$\chi'' + \{L^2 + A(L)\}\chi = 0. \tag{3.6}$$

The vector product of $\boldsymbol{r}$ with (2.16) gives

$$\dot{\boldsymbol{L}} = r^{-2}\{B\hat{\boldsymbol{L}} - C\hat{\omega}\} \tag{3.7}$$

so that

$$L\dot{L} = r^{-2} BL \tag{3.8}$$

or

$$L' = B(L) \tag{3.9}$$

which gives the first integral

$$M = T - \int \frac{\mathrm{d}L}{B(L)}. \tag{3.10}$$

This can be interpreted as an equation defining T in terms of L or L in terms of T. Naturally, if B is zero, the magnitude of the angular momentum is constant.

By virtue of (3.10), (3.6) becomes the by now familiar time-dependent oscillator which characterizes the radial equation for generalized Ermakov systems expressed in the appropriate coordinates.

In like fashion (3.4) is now a three-dimensional non-autonomous first-order system of differential equations. Its structure is suggestive of a time-dependent oscillator written as a system of first-order equations. However, the analogy only helps for a constant L. Before going into the details of the method of solution of (3.4) one comment is appropriate. As a three-dimensional first-order linear system it has three linearly independent solutions. This means that there are in fact three 'Poincaré' vectors.

In view of the geometric interpretation of equations (3.4) some natural examples to consider would be ones which have standard properties of curves. The simplest one is a curve of constant curvature which means that B is zero. The solution of (3.4) is

$$\begin{pmatrix} I \\ J \\ K \end{pmatrix} = \exp\left\{\begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix} T\right\}\begin{pmatrix} I_0 \\ J_0 \\ K_0 \end{pmatrix}$$
$$= \begin{bmatrix} \frac{C}{L\Omega} \\ 0 \\ \frac{L}{\Omega} \end{bmatrix} I_0 + \begin{bmatrix} \frac{L}{\Omega}\sin\Omega T \\ \cos\Omega T \\ -\frac{C}{L\Omega}\sin\Omega T \end{bmatrix} J_0 + \begin{bmatrix} \frac{L}{\Omega}\cos\Omega T \\ -\sin\Omega T \\ -\frac{C}{L\Omega}\cos\Omega T \end{bmatrix} K_0 \tag{3.11}$$

where $\Omega^2 := L^2 + C^2/L^2$ and the scaling has been chosen so that the norm of each vector is one. The standard magnetic monopole has $C = \lambda L$ and is associated with a curve of constant torsion. The usual Poincaré vector of the literature has $I_0 = \Omega$, $J_0 = 0$ and $K_0 = 0$ and is

$$\boldsymbol{P} = \boldsymbol{L} + \lambda\hat{\boldsymbol{r}} \tag{3.12}$$

but we emphasize that there are in fact three vectors.

The solution (3.11) also applies in the case that there is a constant ratio of torsion to curvature, i.e. $C = \lambda L^2$. There is a Poincaré vector $\hat{\boldsymbol{L}} + \lambda\hat{\boldsymbol{r}}$ (*regardless* of A and B) which is time-independent. Then the general solution of (3.4) is written using $\int^T L(\tau)\mathrm{d}\tau$.

Since L is constant the solution of the radial equation (3.6) is simply

$$\chi(T) = E\sin\omega T + F\cos\omega T \tag{3.13}$$

where $\omega^2 := L^2 + A(L)$ and E and F are constants of integration. (We consider only ω real and non-zero. The other two possibilities can be treated in a similar fashion.) From (3.13) and the definition of T we have

$$t = \int \frac{\mathrm{d}T}{(E\sin\omega T + F\cos\omega T)^2} \tag{3.14}$$

which is easily evaluated and inverted to give T in terms of t and, through (3.13), we have

$$r(t) = \left[\frac{1}{E^2+F^2} + \frac{2\omega^2}{F^2}(t-t_0)^2\right]^{1/2}. \tag{3.15}$$

With the general solution of (3.2) inverted to give $\hat{\boldsymbol{r}}$, multiplication by r gives $\boldsymbol{r}(t)$ with six constants of integration and hence the general solution. This solution applies to all problems associated with constant curvature.

We would expect to find three conserved vectors as the form posited for $\boldsymbol{P}$ spans the space (except for exceptional points where degeneracy occurs). One is reminded of the work of Fradkin (1965, 1967) and Yoshida (1987, 1989) on the existence of Laplace–Runge–Lenz vectors for central force and other three-dimensional problems.

The procedure described in detail above for the constant curvature case applies *mutatis mutandis* for the general equation (2.16). Usually the equations become non-autonomous with a consequent increase in the degree of difficulty of solution. This is particularly the case with (3.4) which in the autonomous case is solved by a straightforward exponentiation of the coefficient matrix by time. Nevertheless the general equation can be treated.

By construction $\boldsymbol{P}$ is a constant vector and I, J and K are not independent when the magnitude of $\boldsymbol{P}$ is specified. (In the case of the single vector there is not much point to it, but, when there are three vectors spanning the space, there is no small appeal in specifying unit vectors.) Only two dependent variables are needed and we introduce the transformation (the so-called Weierstrass transformation of Forsyth (1904); see also Kamke (1971)),

$$\xi = \frac{I + \mathrm{i}J}{1 - K} \qquad \eta = -\frac{I + \mathrm{i}J}{1 + K}. \tag{3.16}$$

Together with the normalization of $\boldsymbol{P}$, (3.16) leads to a common differential equation for ξ and η which is of Riccati form, namely

$$w' + \mathrm{i}Lw + \frac{\mathrm{i}L}{2C}(1 - w^2) = 0 \tag{3.17}$$

where w stands for ξ and η in turn. The transformation

$$w = 2\mathrm{i}Cy'/Ly \tag{3.18}$$

yields the linear second-order equation

$$y'' + \left(\frac{C'}{C} - \frac{L'}{L} + \mathrm{i}L\right)y' + \frac{L^2}{4C^2}y = 0 \tag{3.19}$$

which is trivially related via

$$y = \left(\frac{L}{C}\right)^{1/2} u\mathrm{e}^{-\mathrm{i}/2\int L\,\mathrm{d}T} \tag{3.20}$$

to the standard time-dependent harmonic oscillator (TDHO)

$$u'' + \left\{\frac{1}{4}\left(\frac{C'}{C} - \frac{L'}{L} + \mathrm{i}L\right)^2 - \frac{1}{2}\left(\frac{C'}{C} - \frac{L'}{L} + \mathrm{i}L\right)' + \frac{L^2}{4C^2}\right\}u = 0. \tag{3.21}$$

Given the solution for u, ξ and η follow through (3.18) and (3.20). The components of $\boldsymbol{P}$ are given by

$$I = \frac{1 - \xi\eta}{\xi - \eta} \qquad J = \frac{\mathrm{i}(1 + \xi\eta)}{\xi - \eta} \qquad K = \frac{\xi + \eta}{\xi - \eta}. \tag{3.22}$$

Needless to remark the tricky business is always the solution of the TDHO equation (3.21). We illustrate this with what appears to be a fairly innocuous set of functions B and C being proportional to L, i.e.

$$B = \alpha L \qquad C = \beta L. \tag{3.23}$$

Then

$$L = L_0\mathrm{e}^{\alpha T} \tag{3.24}$$

and (3.21) becomes

$$u'' - \tfrac{1}{4}[L_0^2 e^{2\alpha T} + 2i\alpha L_0 e^{\alpha T} - \beta^{-2}]u = 0 \tag{3.25}$$

which is Whittaker's differential equation in slightly disguised form. With the solution to (3.25) the route back to ξ and η via (3.18) and (3.20) is straightforward. To keep things simple we take $A(L)$ to be zero. The solution of the radial equation (3.6) is

$$\chi = EJ_0(L_0 e^{\alpha T}) + FY_0(L_0 e^{\alpha T}) \tag{3.26}$$

where J_0 and Y_0 are Bessel's functions and, as before, E and F are constants. However, the determination of t is via

$$t - t_0 = \int \frac{dT}{\{EJ_0(L_0 e^{\alpha T}) + FY_0(L_0 e^{\alpha T})\}^2} \tag{3.27}$$

for which a closed expression is not known.

4. Some 'weak' considerations

Leach (1991) proposed that systems with Ermakov invariants which did not possess $sl(2, \mathbb{R})$ symmetry should be termed 'weak'. Athorne (1991), although not disagreeing with the distinction, noted that other classifications—such as Hamiltonian and non-Hamiltonian—were also important. Indeed, the point of that letter was that those (non-Hamiltonian) systems described, and which had only *one* global invariant, could be understood as 'linear extensions' of an underlying Hamiltonian system with appropriate choice of time variable. Here we wish to consider a few examples of systems for which only the angular equations possess $sl(2, \mathbb{R})$ symmetry. We maintain $so(3)$ symmetry overall so that the radial equation has the form

$$\ddot{r} - \frac{L^2}{r^3} = \frac{1}{r^3}A(L) + f(r, L) \tag{4.1}$$

where $f(r, L)$ is the symmetry-breaking term. The analysis of the angular equations is the same which means that, in principle, we have $L = L(T)$ and the three Poincaré vectors. In terms of the inverse radial variable χ and new time (4.1) is

$$\chi'' + [A(L) + L^2]\chi + \frac{1}{\chi^2} f\left(\frac{1}{\chi}, L\right) = 0. \tag{4.2}$$

When f is zero, (4.2), as the equation for the TDHO, is transformed to autonomous form by the transformation

$$J = \frac{\chi}{\rho} \qquad \tau = \int \rho(T)^{-2}\, dT \tag{4.3}$$

where ρ is a solution of the Pinney equation (Pinney 1950)

$$\frac{d^2\rho}{dT^2} + [A(L) + L^2]\rho = \rho^{-3}$$

and $L = L(T)$ through (3.10). One could hope that for some functions f that the transformation (4.3) would render it autonomous. For this to happen it is necessary for $\rho = g(L)$ and the argument of f to be $\chi^{-1}g(L)$, where g is a solution to a Pinney-type equation with L as independent variable containing $A(L)$ and $B(L)$.

Such constraints are not required in a few cases. If the additional force is due to a Newton–Cotes potential, (4.2) is as if f were zero and $A(L)$ changed. For a Kepler-type potential

$$f\left(\frac{1}{\chi}, L\right) = \mu(L)\chi^2 \tag{4.5}$$

so that (4.3) is just the non-homogeneous time-dependent oscillator and is solved in standard fashion. For an oscillator type potential

$$f\left(\frac{1}{\chi}, L\right) = \mu(L)\chi^{-1} \tag{4.6}$$

which makes (4.2) a Pinney equation with time-dependent coefficients. If μ is independent of L or $B = 0$, this can be treated as if it were the standard TDHO problem. If such is not the case, the best that one can do is to introduce a time-dependent transformation which converts (4.2) to a generalized Emden–Fowler equation of order -3.

5. Conclusion

In the case that (2.16) has a Hamiltonian representation the Poincaré vectors will have the Lie algebra $so(4)$ under the operation of taking the Poisson Bbracket. The question is under what circumstances does it have a Hamiltonian? (One would not expect the usual Poisson bracket relation $[z_\mu, z_\nu]_{\mathrm{PB}} = J_{\mu\nu}$, ($z_i = q_i, z_{n+1} = p_i$ and J is the $2n \times 2n$ symplectic matrix) but more the monopole type of relation, i.e. seek $H : \dot{q} = [q, H]_{\mathrm{PB}}$ and $\dot{p} = [p, H]_{\mathrm{PB}}$ lead to the equation of motion.) There are two cases of (2.16) to consider: (i) when (2.16) is itself Hamiltonian; and (ii) when (2.16) possesses a global invariant which is not, however, a Hamiltonian function for the system. In the latter case the possibility asrises that this invariant is a Hamiltonian function for a subsystem on an appropriate phase space, as in Athorne (1991).

Acknowledgments

KSG thanks the Foundation for Research Development of South Africa for its support. PGLL expresses his appreciation to Sisters M MacFarlane and S Hucks and their staff of the Orthopaedic Ward of Addington Hospital, Durban, for their tender care while part of the manuscript was written and to an unnameable motorist who made his stay possible.

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